Sobolev stability of shear boundary layers for the steady Navier-Stokes equations (Mathematical Analysis in Fluid and Gas Dynamics)

Maekawa, Yasunori

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Yasunori Maekawa
Department of Mathematics, Kyoto University

1 Introduction

We study the vanishing viscosity limit of the two-dimensional steady Navier-Stokes equations:

$$\begin{cases} v^\nu \cdot \nabla v^\nu - \nu \Delta v^\nu + \nabla q^\nu = g^\nu \,, \quad (x,y) \in \mathbb{T}_\kappa \times \mathbb{R}_+ \,, \\ \operatorname{div} v^\nu = 0 \,, \quad (x,y) \in \mathbb{T}_\kappa \times \mathbb{R}_+ \,, \\ v^\nu|_{y=0} = 0. \end{cases} \tag{1.1}$$

Here $\mathbb{T}_\kappa = \mathbb{R}/(2\pi\kappa)\mathbb{Z}$, $\kappa > 0$, is a torus with periodicity $2\pi\kappa$, $\mathbb{R}_+ = \{y \in \mathbb{R} \mid y > 0\}$, while $v^\nu = (v_1^\nu, v_2^\nu)$ and q^ν are respectively the unknown velocity field and pressure field of the fluid. The positive constant ν is the viscosity coefficient. The vector field g^ν is an external force, decaying fast enough at infinity. The usual no-slip condition is prescribed at $y = 0$.

Understanding the behaviour of v^ν for small ν is a classical and difficult problem: ∇v^ν tends to blow-up near the boundary as $\nu \to 0$, and the analysis of this so-called boundary layer is still a challenging problem. In 1904, L. Prandtl in 1904 suggested asymptotics of the form

$$\begin{aligned} &v^\nu(x,y) \sim \big(V_1(x, y/\sqrt{\nu}), \sqrt{\nu} V_2(x, y/\sqrt{\nu})\big) \quad \text{near the boundary,} \\ &v^\nu(x,y) \sim v^0(x,y) \quad \text{away from the boundary,} \end{aligned} \tag{1.2}$$

where $V = (V_1, V_2)(x, Y)$ depends on a rescaled variable $Y = y/\sqrt{\nu}$. Hence, in the Prandtl model, the boundary layer has a characteristic scale $\sqrt{\nu}$ and it connects to an Euler solution v^0 as $Y \to +\infty$. By plugging the expansion in (1.1), one obtains a kind of reduced Navier-Stokes system on V, the Prandtl equation. As pointed out by Prandtl himself, this formal asymptotics is expected to have a limited range of validity, due to an instability phenomenon called boundary layer separation. This instability is typical of flows around obstacles. Roughly, under an adverse pressure gradient in the boundary layer, past a certain distance $x = x_*$ from the leading edge of the obstacle, the stress $\partial_y v_1^\nu|_{y=0}$ may vanish. This leads to the appearance of a reverse flow for $x > x_*$, and detachment of the boundary layer streamlines; see [28].

Mathematically, the importance of this phenomenon has been well recognized in the analysis of the steady Prandtl model. On one hand, it is known from the works of Oleinik [25] that given a horizontal velocity V_1 at $x = 0$ satisfying $V_1|_{x=0} > 0$, $\partial_Y V_1|_{x=0,Y=0} > 0$, one can construct a local in x smooth solution of the Prandtl equation. This result is based on the so-called Von Mises transform, which turns the Prandtl equation into a nonlinear heat equation, with x as an evolution variable. Moreover, this smooth solution exists as long as $V_1 > 0$ and $\partial_Y V_1|_{Y=0} > 0$. On the other hand, there exists blowing-up solutions: it was established recently in [2], see also [10, 23, 4]. Still, these results leave aside the behaviour of the full system (1.1), and the justification of the Prandtl asymptotics (1.2) prior to separation. In this note we report a recent progress filling in this gap.

It should be stressed that even if the Prandtl equation is successfully solved the verification of the *Prandtl expansion* is highly nontrivial. One reason is the difference of the structure of the pressure, for in the Prandtl model the pressure field is a prescribed quantity, while in the Navier-Stokes model it is an unknown quantity and also the source of nontrivial nonlocality. To understand the fundamental stability/instability mechanism it is therefore a good starting point to study the Prandtl expansion around the shear boundary layer, in which the solvability of the Prandtl equation itself is almost trivial and thus one can focus on the typical stability property of the boundary layer in the level of the Navier-Stokes equations.

Most recent mathematical results on the validity of the Prandtl asymptotics are actually related to the *unsteady* Navier-Stokes equations, even in the case of the shear boundary layer. In such case, it is now well-understood that the justification of the Prandtl approach requires stringent assumptions on the data. The underlying reason is the presence of many hydrodynamic instabilities. Even to hope for short time stability, one must impose either restrictions on the structure of the perturbations [20, 24], or strong regularity assumptions. As regards the well-posedness of the Prandtl model, we refer to [19, 17, 6, 1, 22, 30, 9, 18] and citations therein. As regards the full Navier-Stokes model, a complete justification of the Prandtl theory was obtained for analytic data [26, 27, 29] and for the initial vorticity supported away from the boundary [21, 5]. On the contrary, counterexamples to the H^1 stability of Prandtl expansions of shear flow type was provided by Grenier in [11], using boundary layer profiles with inflexion points. Even in the favourable case of monotonic and concave boundary layer profiles, the boundary layer expansion (1.2) is not stable in a Sobolev framework. This is due to a viscous instability mechanism, the so-called Tollmien-

Schlichting wave. This instability, identified in the first half of the 20th century [3], was examined by Grenier, Guo and Nguyen [12]. Properly rescaled, their analysis provides highly growing eigenmodes of the linearized Navier-Stokes system around a shear flow of Prandtl type. These eigenmodes have high x-frequency$n \sim \nu^{-3/8}$, and associated growth rate $\sigma \sim n^{2/3} \sim \nu^{-1/4}$. For arbitrary small ν, these high frequencies must have very small initial amplitude to be controlled on a time scale independent of ν: namely, one can only hope for a short time stability result in functional spaces of Gevrey class $3/2$ in x. A result in this direction was obtained recently by the authors and N. Masmoudi in [8]. In fact, the paper [8] is the first contribution that justifies the Prandtl expansion for given data strictly below the real analytic regularity under the presence of nontrivial high frequency instability. See [13] for related statements.

Less is known in the steady case. However, the analysis of the Tollmien-Schlichting wave in the literature indicates that that the high frequency instability is in fact strongly connected with the time frequency. Thus, there is a good hope in the steady case to achieve the stability in the Sobolev framework. This note reports that indeed the linearization around the shear boundary layer $U^\nu(x,y) = (U_s(y/\sqrt{\nu}), 0)$ can be well analyzed when $U_s(Y) > 0$ for $Y > 0$, $U_s(0) = 0$, and $U_s'(0) > 0$, resulting in the nonlinear stability under suitable assumptions for perturbations. The above conditions on the shear boundary layer U are somehow minimal in view of the previous discussion: they forbid reverse flow and boundary layer separation. Related to the result of this note, the reader is referred to [15, 16] in the steady case but under the inhomogeneous Dirichlet conditions. For instance, Guo and Nguyen consider in [15] the steady Navier-Stokes equations in a half-plane, with a positive Dirichlet datum for the horizontal velocity. They construct general boundary layer expansions for this problem and prove their Sobolev stability through the use of original energy functionals. Although the result stated in this note is only around the shear boundary layer, the Prandtl expansion around general boundary layer in the steady case is recently established in [14]; see Remark 3.

2 Main result

Let $U_s = U_s(Y) \in C^2(\overline{\mathbb{R}_+})$ such that

$$U_s(0) = 0\,, \qquad U_s > 0 \;\text{ in } Y > 0\,, \qquad \lim_{Y\to\infty} U_s(Y) = U_E > 0\,, \tag{2.1}$$

$$\partial_Y U_s(0) > 0\,, \tag{2.2}$$

$$\sum_{k=1,2} \sup_{Y\geq 0} (1+Y)^3 |\partial_Y^k U_s(Y)| < \infty\,. \tag{2.3}$$

From the continuity and (2.2) we have $\partial_Y U_s > 0$ on $0 \le Y \le 4Y_0$ for some $Y_0 \in (0,1]$. This nondegeneracy near the boundary will be crucial. We then consider the shear flow

$$U^\nu = (U_s^\nu(y), 0)\,, \qquad U_s^\nu(y) = U_s(y/\sqrt{\nu}). \tag{2.4}$$

Obviously, (2.4) can be seen as a solution of (1.1), setting $g^\nu = -\nu\partial_y^2 U^\nu$ and $q^\nu = 0$. The goal of the paper is to establish stability estimates for this solution of boundary layer type. Denoting $u^\nu = v^\nu - U^\nu$ the perturbation induced by $f^\nu = g^\nu + \nu\partial_y^2 U^\nu$, we get

$$\begin{cases} U_s^\nu \partial_x u^\nu + u_2^\nu \partial_y U_s^\nu \mathbf{e}_1 - \nu\Delta u^\nu + \nabla p^\nu = -u^\nu \cdot \nabla u^\nu + f^\nu\,, \quad (x,y) \in \mathbb{T}_\kappa \times \mathbb{R}_+\,, \\ \operatorname{div} u^\nu = 0\,, \quad (x,y) \in \mathbb{T}_\kappa \times \mathbb{R}_+\,, \\ u^\nu|_{y=0} = 0. \end{cases} \tag{2.5}$$

Here $\mathbf{e}_1 = (1,0)$. We then have to specify a functional setting, with $2\pi\kappa$ periodicity in x. Let $\mathcal{P}_n$, $n \in \mathbb{Z}$, be the orthogonal projection on the n-th Fourier mode in variable x:

$$(\mathcal{P}_n u)(x,y) = u_n(y) e^{i\tilde{n}x}\,, \qquad \tilde{n} = \frac{n}{\kappa}\,, \qquad u_n(y) = \frac{1}{2\pi\kappa}\int_0^{2\pi\kappa} u(x,y) e^{-i\tilde{n}x} dx\,, \tag{2.6}$$

The divergence-free and homogeneous Dirichlet conditions imply $u_0 = (u_{0,1}, 0)$. Setting

$$\mathcal{Q}_0 u = (I - \mathcal{P}_0) u\,, \tag{2.7}$$

where I is the identity operator, we can identify u with the couple $(u_{0,1}, \mathcal{Q}_0 u)$. With this identification we introduce

$$\begin{aligned} X = \Big\{ &(u_{0,1}, \mathcal{Q}_0 u) \in BC(\overline{\mathbb{R}_+}) \times W_0^{1,2}(\mathbb{T}_\kappa \times \mathbb{R}_+)^2 \mid \quad \partial_y u_{0,1} \in L^2(\mathbb{R}_+)\,, \qquad u_{0,1}|_{y=0} = 0\,, \\ &\|u\|_X = \|u_{0,1}\|_{L^\infty(\mathbb{R}_+)} + \|\partial_y u_{0,1}\|_{L^2(\mathbb{R}_+)} + \sum_{n \neq 0} \|u_n\|_{L^\infty(\mathbb{R}_+)} + \|\mathcal{Q}_0 u\|_{W^{1,2}(\mathbb{T}_\kappa \times \mathbb{R}_+)} < \infty \Big\}\,, \end{aligned} \tag{2.8}$$

where the Sobolev space $W_0^{1,2}(\mathbb{T}_\kappa \times \mathbb{R}_+)$ is defined as the subspace of $W^{1,2}(\mathbb{T}_\kappa \times \mathbb{R}_+)$ with functions having the zero boundary trace on $y = 0$. For simplicity we assume that $f^\nu = \mathcal{Q}_0 f^\nu$ below, though it is not difficult to extend our result to a general case by imposing a suitable condition on $f_0^\nu(y)$.

Theorem 1 ([7]). *There exist positive numbers $\kappa_0, \nu_0, \epsilon$ such that the following statement holds for $0 < \kappa \le \kappa_0$ and $0 < \nu \le \nu_0$. If $f^\nu = \mathcal{Q}_0 f^\nu$ and $\|f^\nu\|_{L^2} \le \epsilon \nu^{\frac{3}{4}} |\log \nu|^{-1}$ then there exists a unique solution $(u^\nu, \nabla p^\nu) \in \big(X \cap W_{loc}^{2,2}(\mathbb{T}_\kappa \times \mathbb{R}_+)^2\big) \times L^2(\mathbb{T}_\kappa \times \mathbb{R}_+)^2$ to (2.5) such that*

$$\begin{aligned} &\|u_{0,1}^\nu\|_{L^\infty} + \nu^{\frac{1}{4}} \|\partial_y u_{0,1}^\nu\|_{L^2} \\ &\qquad + \sum_{n \neq 0} \|u_n^\nu\|_{L^\infty} + \nu^{-\frac{1}{4}} \|\mathcal{Q}_0 u^\nu\|_{L^2} + \nu^{\frac{1}{4}} \|\nabla \mathcal{Q}_0 u^\nu\|_{L^2} \le \frac{C|\log \nu|^{\frac{1}{2}}}{\nu^{\frac{1}{4}}} \|f^\nu\|_{L^2}\,, \end{aligned} \tag{2.9}$$

Here C is independent of ν and κ.

Remark 1. The main structural assumptions of our stability theorems are (2.1) and (2.2), which are natural in view of the previous comments on boundary layer separation. Another important requirement is the smallness condition on κ: it means that our stability result is only local in space.

Remark 2. The perturbation u^ν converges to a constant shear flow at infinity:

$$\lim_{y\to+\infty} v^\nu = (c^\nu, 0). \tag{2.10}$$

First, the requirement $\mathcal{Q}_0 u^\nu \in W^{1,2}$ implies that $\mathcal{Q}_0 u^\nu$ goes to zero at infinity. Then, as regards the x-average $u_0^\nu = (u_{0,1}^\nu, 0)$, we deduce from the first line of (2.5) and the fact that $f_0^\nu = 0$:

$$-\nu\partial_y^2 u_{0,1}^\nu = -\partial_y(Q_0 u_2^\nu Q_0 u_1^\nu)_0.$$

As $\partial_y u_{0,1} \in L^2$, we can integrate this identity from $y = +\infty$ to deduce

$$-\nu\partial_y u_{0,1}^\nu = -(Q_0 u_2^\nu Q_0 u_1^\nu)_0$$

Eventually, as the right-hand side belongs to L^1, we find (2.10) with

$$c^\nu = \frac{1}{\nu}\int_{\mathbb{R}_+} (Q_0 u_2^\nu Q_0 u_1^\nu)_0.$$

Note that this constant at infinity can not be prescribed. Moreover, it obeys the bound

$$|c^\nu| \le \frac{C|\log\nu|^{\frac{1}{2}}}{\nu^{\frac{1}{4}}}\|f^\nu\|_{L^2}$$

as a consequence of estimate (2.9).

Remark 3. Just after our manuscript submission on the arXiv, Y. Guo and S. Iyer have submitted the very interesting preprint [14]. They establish there the Sobolev stability of a subclass of Prandtl expansions, the main example of which being the famous Blasius flow.

3 Key estimate for linearization

The core of the proof of Theorem 1 is the analysis of the linearized system around U^ν. Through a Fourier transform in x, it can be written

$$\begin{cases} i\tilde{n}U_s^\nu u_n + u_{n,2}(\partial_y U_s^\nu)\mathbf{e_1} - \nu(\partial_y^2 - \tilde{n}^2)u_n + \begin{pmatrix} i\tilde{n}p_n \\ \partial_y p_n \end{pmatrix} = f_n\,, \qquad y > 0\,, \\ i\tilde{n}u_{n,1} + \partial_y u_{n,2} = 0\,, \qquad y > 0\,, \\ u_n|_{y=0} = 0\,. \end{cases} \tag{3.1}$$

We remind that $u_n = u_n(y)$ is the n-th Fourier coefficient of the velocity, and $\tilde{n} = n/\kappa$. Note that $|\pm\tilde{1}|$ is large if κ is small. The zero mode does not raise any difficulty. The difficult part is the derivation of good bounds for $\tilde{n} \neq 0$. For κ small enough, we can always ensure that $|\tilde{n}| \gg 1$ for all n. Nevertheless, as $\nu \ll 1$, the tangential diffusion term $-\nu\tilde{n}^2 u_n$ in the first line of (3.1) is in general far too small to control the stretching term $u_{n,2}\partial_y U_s^\nu = O(\frac{1}{\sqrt{\nu}}|u_n|)$. The key result to (3.1) is stated as follows.

Theorem 2 ([7]). *There exist positive numbers κ_0, ν_0, and δ_* such that the following statement holds for any $0 < \kappa \le \kappa_0$, $0 < \nu \le \nu_0$, and $\tilde{n} \neq 0$. For any $f_n \in L^2(\mathbb{R}_+)^2$ there exists a unique solution $u_n \in H^2(\mathbb{R}_+)^2 \cap H_0^1(\mathbb{R}_+)^2$ to (3.1) satisfying the estimates stated below:*
(i) *if $0 < |\tilde{n}| \le \nu^{-\frac{3}{7}}$ then*

$$\|u_n\|_{L^2} \le \begin{cases} \dfrac{C}{|\tilde{n}|^{\frac{1}{2}}}\|f_n\|_{L^2}\,, & 0 < |\tilde{n}| \le \nu^{-\frac{3}{8}} \\ \dfrac{C}{|\tilde{n}|^{\frac{11}{6}}\nu^{\frac{1}{2}}}\|f_n\|_{L^2}\,, & \nu^{-\frac{3}{8}} \le |\tilde{n}| \le \nu^{-\frac{3}{7}}\,, \end{cases} \tag{3.2}$$

$$\|u_n\|_{L^\infty} \le \frac{C}{|\tilde{n}|^{\frac{1}{2}}\nu^{\frac{1}{4}}}\|f_n\|_{L^2}\,, \tag{3.3}$$

$$\|\partial_y u_n\|_{L^2} + |\tilde{n}|\|u_n\|_{L^2} \le \frac{C}{|\tilde{n}|^{\frac{1}{3}}\nu^{\frac{1}{2}}}\|f_n\|_{L^2}\,. \tag{3.4}$$

(ii) *if $\nu^{-\frac{3}{7}} \le |\tilde{n}| \le \delta_*\nu^{-\frac{3}{4}}$ then*

$$\|u_n\|_{L^2} \le \frac{C}{|\tilde{n}|^{\frac{2}{3}}}\|f_n\|_{L^2}\,, \tag{3.5}$$

$$\|\partial_y u_n\|_{L^2} + |\tilde{n}|\|u_n\|_{L^2} \le \frac{C}{|\tilde{n}|^{\frac{1}{3}}\nu^{\frac{1}{2}}}\|f_n\|_{L^2}\,. \tag{3.6}$$

(iii) *if $|\tilde{n}| \ge \delta_*\nu^{-\frac{3}{4}}$ then*

$$\|u_n\|_{L^2} \le \frac{C}{|\tilde{n}|^2\nu}\|f_n\|_{L^2}\,, \tag{3.7}$$

$$\|\partial_y u_n\|_{L^2} + |\tilde{n}|\|u_n\|_{L^2} \le \frac{C}{|\tilde{n}|\nu}\|f_n\|_{L^2}\,. \tag{3.8}$$

As stated in Theorem 2, we distinguish between two regimes: $|\tilde{n}| \ll \nu^{-3/4}$ and $|\tilde{n}| \gtrsim \nu^{-3/4}$. The regime $|\tilde{n}| \gtrsim \nu^{-3/4}$ is not so difficult in virtue of the strong dissipation due to the viscosity, and the direct analysis of system (3.1) is possible.

Stability in the regime $|\tilde{n}| \ll \nu^{-3/4}$ is the most delicate to obtain. It is deduced from a careful analysis of the steady Orr-Sommerfeld system, which is a reformulation of (3.1) in terms of the stream function and of the rescaled variable $Y = y/\sqrt{\nu}$. It reads

$$\begin{cases} OS[\phi] := U_s(\partial_Y^2 - \alpha^2)\phi - U_s''\phi + i\varepsilon(\partial_Y^2 - \alpha^2)^2\phi = -f_2 - \frac{i}{\alpha}\partial_Y f_1\,, & Y > 0\,, \\ \phi|_{Y=0} = \partial_Y\phi|_{Y=0} = 0\,. \end{cases}$$

where parameters α and ε are related to the tangential frequency $\tilde{n}$ and the viscosity ν: $\alpha = \tilde{n}\sqrt{\nu}$ and $\varepsilon = 1/\tilde{n}$. In short, the regime $|\tilde{n}| \ll \nu^{-3/4}$ corresponds to the case $\varepsilon^{1/3}\alpha \ll 1$.

The point is that we are not able to get direct estimates on this system. Instead, we construct the solution through an iterative process, reminiscent of splitting methods in numerical analysis. More precisely, one main idea is to construct a solution to the Orr-Sommerfeld equation in the form of a series, where successive corrections solve alternatively:

- inviscid approximations of the equation, based on the so-called *Rayleigh equation.*
- viscous approximations of the equation, based on the so-called *Airy equation.*

This idea of a splitting method was already present in our Gevrey stability study of the unsteady case [8], and found its origin in article [12]: the construction of an unstable eigenmode for the linearized Navier-Stokes equations was performed with a similar iteration, although more explicit and specific to a narrower regime of parameters. Here and in [8], the convergence of the iteration is rather shown by energy arguments, and adapted to the whole range $|\tilde{n}| \ll \nu^{-3/4}$. But in the steady setting considered here, we must rely on estimates that are totally different from the ones in [8], in order to reach Sobolev stability. Moreover, the implementation of the splitting method is different.

In the inviscid approximation we employ the equation $Ray[\varphi] = f$, where the Rayleigh operator

$$Ray := U_s(\partial_Y^2 - \alpha^2) - U_s'' \tag{3.9}$$

corresponds to neglecting the diffusion in the Orr-Sommerfeld operator. Due to the degeneracy of U_s at $Y = 0$, the derivation of good bounds is found to be delicate. The most difficult case is when $\alpha \ll 1$: indeed taking $\alpha \to 0$ in the Rayleigh equation yields a singular perturbation problem. A crucial point here is that the singularity shows up only when the source term f has nonzero average in Y. Below we use the notation $\|f\| = \|f\|_{L^2_Y(\mathbb{R}_+)}$.

Proposition 1 (Solvability of Rayleigh equation)**.** *Let $f/U_s \in L^2(\mathbb{R}_+)$. Then there exists a unique solution $\varphi \in H^2(\mathbb{R}_+) \cap H^1_0(\mathbb{R}_+)$ to*

$$\begin{cases} Ray[\varphi] = f\,, & Y > 0\,, \\ \varphi|_{Y=0} = 0\,, \end{cases} \tag{3.10}$$

such that
(i) *when* $\alpha \geq 1$,

$$\|\partial_Y \varphi\| + \alpha\|\varphi\| \leq C \min\{\|\frac{Y}{U_s} f\|, \frac{1}{\alpha}\|\frac{f}{U_s}\|\}, \tag{3.11}$$

$$\|(\partial_Y^2 - \alpha^2)\varphi\| \leq C \min\{\|\frac{Y}{U_s} f\|, \frac{1}{\alpha}\|\frac{f}{U_s}\|\} + \|\frac{f}{U_s}\|. \tag{3.12}$$

(ii) *when* $0 < \alpha \leq 1$, *if* $(1+Y)\sigma[f] \in L^2(\mathbb{R}_+)$ *with* $\sigma[f](Y) = \int_Y^\infty f\, dY_1$ *in addition,*

$$\alpha\|\varphi\| \leq C\alpha\|(1+Y)\sigma[f]\| + \frac{C}{\alpha^{\frac{1}{2}}}|\int_0^\infty f\, dY|, \tag{3.13}$$

$$\|\partial_Y \varphi\| \leq C\Big(\|(1+Y)\sigma[f]\| + \|f\|\Big) + \frac{C}{\alpha}|\int_0^\infty f\, dY|, \tag{3.14}$$

$$\|(\partial_Y^2 - \alpha^2)\varphi\| \leq C\Big(\|(1+Y)\sigma[f]\| + \|\frac{f}{U_s}\|\Big) + \frac{C}{\alpha}|\int_0^\infty f\, dY|. \tag{3.15}$$

After the inviscid analysis one needs to collect various estimates on viscous equations of Airy type: they all involve the operator

$$Airy := U_s + i\varepsilon(\partial_Y^2 - \alpha^2). \tag{3.16}$$

The Airy operator is essentially the sum of the Laplacian and the convection. Due to the absence of the stretching term the analysis of the Airy equation $Airy[\psi] = f$ is not so difficult.

Proposition 2 (Solvability of Airy equation)**.** *Let* $f \in L^2(\mathbb{R}_+)$. *Then there exists a unique solution* $\psi \in H^2(\mathbb{R}_+) \cap H_0^1(\mathbb{R}_+)$ *to*

$$\begin{cases} Airy[\psi] = \varepsilon f, & Y > 0, \\ \psi|_{Y=0} = 0, \end{cases} \tag{3.17}$$

such that

$$\|U_s\psi\| + \varepsilon^{\frac{1}{6}}\|\sqrt{U_s}\psi\| + \varepsilon^{\frac{1}{3}}\|\psi\| + \varepsilon^{\frac{2}{3}}\big(\|\partial_Y\psi\| + \alpha\|\psi\|\big) + \varepsilon\|(\partial_Y^2 - \alpha^2)\psi\| \leq C\varepsilon\|f\|, \tag{3.18}$$

and also

$$\|U_s Y\psi\| \leq C\varepsilon\|Yf\| + C\varepsilon^{\frac{4}{3}}\|f\| \tag{3.19}$$

if $(1+Y)f \in L^2(\mathbb{R}_+)$ *in addition. Moreover, if* f *is replaced by* $\partial_Y f$ *or* $\frac{f}{Y}$, *then*

$$\varepsilon^{\frac{1}{2}}\|\sqrt{U_s}\psi\| + \varepsilon^{\frac{2}{3}}\|\psi\| + \varepsilon\big(\|\partial_Y\psi\| + \alpha\|\psi\|\big) \leq C\varepsilon\|f\|. \tag{3.20}$$

In the case when f *is replaced by* $\frac{f}{Y}$ *we also have*

$$\|U_s\psi\| \leq C\varepsilon^{\frac{2}{3}}\|f\|. \tag{3.21}$$

In Proposition 2 the power $\varepsilon^{\frac{1}{3}}$ naturally appears due to the balance between $i\varepsilon\partial_Y^2$ and $U_s \sim Y\partial_Y U_s(0)$ near the boundary. Note that the Rayleigh and Airy operators are naturally involved within the full Orr-Sommerfeld operator through the identities, which are the key in performing the effective iteration:

$$
\begin{aligned}
OS[\phi] &= Ray[\phi] + i\varepsilon(\partial_Y^2 - \alpha^2)^2\phi = Ray[\phi] + i\varepsilon(\partial_Y^2 - \alpha^2)[\frac{1}{U_s}Ray[\phi] + \frac{U_s''}{U_s}\phi], \\
OS[\phi] &= (\partial_Y^2 - \alpha^2)Airy[\phi] - 2\partial_Y(U_s'\phi), \\
OS[\phi] &= Airy[\frac{1}{U_s}Ray[\phi]] + i\varepsilon(\partial_Y^2 - \alpha^2)\frac{U_s''}{U_s}\phi
\end{aligned}
$$

These identities are at the basis of the splitting method alluded to above, which provides a solution to the Orr-Sommerfeld equation under the form of a converging series. This construction is called the Rayleigh-Airy iteration. In this process, a special attention is paid to the possible singularity generated by the Rayleigh equation when $\alpha \ll 1$, which could forbid the convergence of the series. In short, one has to ensure that each "Rayleigh step" is performed with a zero average source term. This major difficulty is new compared to the unsteady analysis in [8], and leads to a different iteration.

Moreover, the Rayleigh-Airy iteration is not enough to conclude: it provides a solution to the Orr-Sommerfeld equation with a given source term, but this solution does not satisfy both Dirichlet and Neumann conditions. Only the Dirichlet condition is maintained through the iteration. One must then combine it with two solutions of the homogeneous Orr-Sommerfeld equation (with an inhomogeneous Dirichlet condition $\phi|_{Y=0} = 1$). These special solutions ϕ_{slow} and ϕ_{fast} are called slow and fast modes, following a terminology of [12].

Proposition 3 (Construction of slow mode)**.** *Let* $0 < \varepsilon \ll 1$ *and* $0 < \alpha \leq 1$*. Then there exists a solution* $\phi_{slow} \in H^4(\mathbb{R}_+)$ *to* $OS[\phi_{slow}] = 0$ *satisfying the following properties:* $\phi_{slow} = \frac{c_E}{\alpha}U_s e^{-\alpha Y} + \phi_{slow,re}$*, where*

$$
\phi_{slow}(0) = 1\,, \tag{3.22}
$$

and

$$
\|\partial_Y\phi_{slow,re}\| + \alpha\|\phi_{slow,re}\| \leq C(\frac{\varepsilon^{\frac{1}{3}}}{\alpha} + 1)\,, \tag{3.23}
$$

$$
\|\partial_Y\phi_{slow,re}\|_{L^\infty} \leq C(\frac{\varepsilon^{\frac{1}{12}}}{\alpha} + \frac{1}{\varepsilon^{\frac{1}{4}}})\,, \tag{3.24}
$$

$$
\|(\partial_Y^2 - \alpha^2)\phi_{slow,re}\| \leq C(\frac{1}{\varepsilon^{\frac{1}{6}}\alpha} + \frac{1}{\varepsilon^{\frac{1}{3}}})\,. \tag{3.25}
$$

In particular, we have

$$\partial_Y \phi_{slow}(0) = \frac{c_E U_s'(0)}{\alpha} + O(\frac{\varepsilon^{\frac{1}{12}}}{\alpha} + \frac{1}{\varepsilon^{\frac{1}{4}}})\,. \tag{3.26}$$

Here c_E *is a number satisfying the asymptotics* $c_E = \frac{\partial_Y U_s(0)}{U_E^2} + O(\alpha)$ *for* $0 < \alpha \ll 1$.

Proposition 4 (Construction of fast mode)**.** *There exists a positive number* δ_1 *such that if* $0 < \varepsilon \ll 1$ *and* $\varepsilon^{\frac{1}{3}}\alpha \le \delta_1$ *then there exists a function* $\phi_{fast} \in H^4(\mathbb{R}_+)$ *satisfying* $OS[\phi_{fast}] = 0$ *and*

$$\|\partial_Y \phi_{fast}\| + \alpha\|\phi_{fast}\| \le \frac{C}{\varepsilon^{\frac{1}{6}}}\,, \tag{3.27}$$

$$\|(\partial_Y^2 - \alpha^2)\phi_{fast}\| \le \frac{C}{\varepsilon^{\frac{1}{2}}}\,, \tag{3.28}$$

and also

$$\phi_{fast}(0) = 1\,, \tag{3.29}$$

$$\partial_Y \phi_{fast}(0) = \Big(e^{\frac{\pi}{6}i} U_s'(0)^{\frac{1}{3}} 3^{-\frac{2}{3}} \Gamma(\frac{1}{3}) + O(\varepsilon^{\frac{1}{3}}\alpha) + O(\varepsilon^{\frac{1}{3}})\Big)\varepsilon^{-\frac{1}{3}}\,. \tag{3.30}$$

Here $\Gamma(s)$ *is the Gamma function.*

Let us stress that the construction of the slow and fast modes can not be performed in an abstract way, like for the solution coming from the Rayleigh-Airy iteration. They are rather obtained starting from an explicit approximation (of inviscid type for the slow mode, of viscous "boundary layer type" for the fast mode), which fulfills the inhomogeneous condition, but solves approximately the equation. One can then add a corrector to get an exact solution, notably making use of the Rayleigh-Airy iteration developed earlier. The important point is to show the nondegenerate property

$$\det \begin{pmatrix} \phi_{slow}(0) & \phi_{fast}(0) \\ \partial_Y \phi_{slow}(0) & \partial_Y \phi_{fast}(0) \end{pmatrix} \ne 0,$$

which enables us to recover the noslip boundary condition for the Orr-Sommerfel equation. The proof of the linear stability result in the regime $|\tilde{n}| \ll \nu^{-3/4}$ is then achieved.

Once the linear estimates of Theorem 2 are shown, the proof of our main Theorem 1 can be completed classically by a fixed point argument.

References

[1] Alexandre, R., Wang, Y.-G., Xu, C.-J., and Yang, T.; Well-posedness of the Prandtl equation in Sobolev spaces. J. Amer. Math. Soc., 28(3):745-784, 2015.

[2] Dalibard, A.-L., and Masmoudi, N.; Phénomène de séparation pour l'équation de Prandtl. Séminaire Laurent Schwartz, Exp IX, 18. Ed. Ecole Polytechnique, 2016.

[3] Drazin, P. G., and Reid,W. H.; *Hydrodynamic stability.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2004. With a foreword by John Miles.

[4] E, W.; Boundary layer theory and the zero-viscosity limit of the Navier-Stokes equation. Acta Math. Sin. 16, 2 (2000) 207-218.

[5] Fei, M., Tao, T., and Zhang, Z.; On the zero-viscosity limit of the Navier-Stokes equations in $\mathbb{R}^3_+$ without analyticity. J. Math. Pures Appl., 112:170-229, 2018.

[6] Gerard-Varet, D., and Dormy, E.; On the ill-posedness of the Prandtl equation. J. Amer. Math. Soc., 23(2):591-609, 2010.

[7] Gerard-Varet, D., and Maekawa, Y.; Sobolev stability of Prandtl expansions for the steady Navier-Stokes equations. arXiv:1805.02928.

[8] Gerard-Varet, D., Maekawa, Y., and Masmoudi, N.; Gevrey stability of Prandtl expansions for 2D Navier-Stokes flows. Duke Math. J. 167 (13), 2531-2631, 2018.

[9] Gerard-Varet, D., and Masmoudi, N.; Well-posedness for the Prandtl system without analyticity or monotonicity. Ann. Scient. Ec. Norm. Sup., 48(4):1273-1325, 2015.

[10] Goldstein, S.; On laminar boundary layer flow near a position of separation. Quaterly J. Mech. Applied Math. 1, 43-69, 1948.

[11] Grenier, E.; On the nonlinear instability of Euler and Prandtl equations. Comm. Pure Appl. Math., 53(9):1067-1091, 2000.

[12] Grenier, E., Guo, Y., and Nguyen, T.; Spectral instability of characteristic boundary layer flows. Duke Math. J., 165:3085-3146, 2016.

[13] Grenier, E., and Nguyen, T.; Sharp bounds on linear semigroup of Navier-Stokes with boundary layer norms. Preprint arXiv, 2017.

[14] Guo, Y., Iyer, S.; Validity of steady Prandtl layer expansions. arXIv:1805.05891, May 2018.

[15] Guo, Y., Nguyen, T.; Prandtl boundary layer expansions of steady Navier-Stokes flows over a moving plate. Ann. PDE 3 (2017), no. 1, Art. 10, 58 pp.

[16] Iyer, S.; Steady Prandtl boundary layer expansions over a rotating disk. Arch. Ration. Mech. Anal. 224 (2017), no. 2, 421-469.

[17] Kukavica, I., and Vicol, V.; On the local existence of analytic solutions to the Prandtl boundary layer equations. *Commun. Math. Sci.*, 11(1):269-292, 2013.

[18] Li, W.-X., and Yang., T.; Well-posedness in Gevrey space for the Prandtl equations with non-degenerate critical points To appear in J. Eur. Math. Soc.

theory, to appear in CPAM.

[19] Lombardo, M. C., Cannone, M., and Sammartino, M.; Well-posedness of the boundary layer equations. SIAM J. Math. Anal., 35(4):987-1004, 2003.

[20] Lopes Filho, M. C., Mazzucato, A. L., and Nussenzveig Lopes H. J.; Vanishing viscosity limit for incompressible flow inside a rotating circle. Phys. D, 237(10-12):1324-1333, 2008.

[21] Maekawa, Y.; On the inviscid limit problem of the vorticity equations for viscous incompressible flows in the half plane, Comm. Pure and Applied Math., 67:1045-1128, 2014.

[22] Masmoudi, N., and Wong, T. K.; Local-in-time existence and uniqueness of solutions to the Prandtl equations by energy methods. *Comm. Pure Appl. Math.*, 68(10):1683-1741, 2015.

[23] Matsui, S., and Shirota, T.; On separation points of solutions to Prandtl boundary layer problem. Hokkaido Math. J., 13(1): 92-108, 1984.

[24] Mazzucato, A. L., and Taylor, M. E.; Vanishing viscosity plane parallel channel flow and related singular perturbation problems. Analysis & PDE, 1(1):35-93, 2008.

[25] Oleinik, O. A., and Samokhin.,V. N.; *Mathematical models in boundary layer theory*, volume 15 of Applied Mathematics and Mathematical Computation. Chapman & Hall/CRC, Boca Raton, FL, 1999.

[26] Sammartino, M., and Caflisch, R. E.; Zero viscosity limit for analytic solutions, of the Navier-Stokes equation on a half-space. I. Existence for Euler and Prandtl equations. *Comm. Math. Phys.*, 192(2):433-461, 1998.

[27] Sammartino, M., and Caflisch, R. E.; Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. II. Construction of the Navier-Stokes solution. *Comm. Math. Phys.*, 192(2):463-491, 1998.

[28] Schlichting, H. *Boundary layer theory.* Eighth edition. Springer-Verlag, Berlin, 2000.

[29] Wang, C., Wang, Y., and Zhang, Z.; Zero-viscosity Limit of the Navier-Stokes equations in the analytic setting. Arch. Rational. Mech. Anal. 224 (2017) 555-595.

[30] Xin, Z., and Zhang., L.; On the global existence of solutions to the Prandtl's system. Adv. Math., 181:88-133, 2004.

Department of Mathematics
Kyoto University
Kyoto 606-8502
JAPAN
E-mail address: maekawa@math.kyoto-u.ac.jp

京都大学・数学教室　　前川　泰則